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Odsjek: Matematika i informatika
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Pismeni ispit iz Linearne algebre

Pravila: Svaku formulu koju mislite koristit, u sva 4 zadatka, obavezno napisati, kao i značenja simbola iz formule. Ispit pisati isključivo hemiskom olovkom plave ili crne tinte. Prije rješenja prepisati postavku (tekst) zadatka.

1. Neka je $\mathcal{V}$ vektorski prostor svih matrica oblika 2×2 nad poljem realnih brojeva. Neka je $\mathcal{W}_1$ skup matrica oblika

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix}$$

a neka je $\mathcal{W}_2$ skup svih matrica oblika

$$\begin{pmatrix} a & b \\ -a & c \end{pmatrix}$$

(a) Dokazati da su $\mathcal{W}_1$ i $\mathcal{W}_2$ podprostori od $\mathcal{V}$.
(b) Odrediti bazu i dimenziju od $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_1 + \mathcal{W}_2$ i $\mathcal{W}_1 \cap \mathcal{W}_2$.

2. Zadan je linearni operator $T : \mathrm{Mat}_{2\times2}(\mathbb{R}) \longrightarrow \mathrm{Mat}_{2\times2}(\mathbb{R})$ sa

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-b & -a+b+2c \\ a-c-d & -a+2c+d \end{bmatrix}.$$

(a) Odrediti po jednu bazu za $\ker(T)$ i $\mathrm{im}(T)$.
(b) Odredite matricu koordinata od T u odnosu na standardnu bazu prostora $\mathrm{Mat}_{2\times2}(\mathbb{R})$.

3. Posmatrajmo realni unitarni prostor $\mathcal{P}_2$, gdje za polinome

$$p = p(x) = p_0 + p_1 x + p_2 x^2 \quad \text{i} \quad q = q(x) = q_0 + q_1 x + q_2 x^2$$

je definisan unutrašnji proizvod na sljdeći način

$$\langle p, q \rangle = p_0 q_0 + p_1 q_1 + p_2 q_2.$$

Provjeriti da li su polinomi

$$u_1 = 3 + 4x + 5x^2, \quad u_2 = 9 + 12x + 5x^2, \quad u_3 = 1 - 7x + 25x^2,$$

linearno nezavisni u $\mathcal{P}_2$, pa pomoću njih formirati ortonormiranu bazu za $\mathcal{P}_2$.

4. Neka je $\mathcal{M}$ podprostor unitarnog prostora $\mathrm{Mat}_{2\times2}(\mathbb{R})$ generisan matricama $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ i $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$.
Odredite jednu bazu za ortogonalni komplement od $\mathcal{M}$, te prikažite matricu $X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ u obliku $X = Y_1 + Y_2$, gdje je $Y_1 \in \mathcal{M}$, a $Y_2 \in \mathcal{M}^{\perp}$. (Standardni skalarni proizvod u $\mathrm{Mat}_{2\times2}(\mathbb{R})$ je $\langle A, B \rangle = \mathrm{trag}(AB^{\top})$).

(#) Neka je V vektorski prostor svih matrica oblika 2×2 nad poljem realnih brojeva. Neka je W_1 skup matrica oblika

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix}$$

a neka je W_2 skup matrica oblika

$$\begin{pmatrix} a & b \\ -a & c \end{pmatrix}$$

(a) Dokazati da su W_1 i W_2 podprostori od V.

(b) Odrediti bazu i dimenziju od W_1, W_2, W_1+W_2 i $W_1 \cap W_2$.

Rj:

(a) Prema definiciji, W je podprostor vektorskog prostora V akko je W neprazan skup i ako vrijedi

(A1) $A, B \in W \Rightarrow A+B \in W$

(M1) $A \in W \Rightarrow \alpha A \in W$ za $\forall \alpha \in \mathbb{R}$

Imajući ovu tvrdnju na vidu, dokaz da su W_1 i W_2 podprostori je lagan, i ostavljamo ga za vježbu.

(b) $$W_1 = \left\{ \begin{pmatrix} x & -x \\ y & z \end{pmatrix} \middle| \; x,y,z \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \cdot x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z \middle| \; x,y,z \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Kako su ovi vektori linearni nezavisni baza za W_1 je $\left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ a dimenzija je 3.

Slično

$$\mathcal{W}_2 = \left\{ \begin{pmatrix} a & b \\ -a & c \end{pmatrix} \middle| a,b,c \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} a + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} b + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} c \middle| a,b,c \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Kako su ovi vektori linearno nezavisni, baza ~~za~~ za $\mathcal{W}_2$ je

$\left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ a dimenzija je 3.

$$\mathcal{W}_1 + \mathcal{W}_2 = \{ W_1 + W_2 \mid W_1 \in \mathcal{W}_1 \text{ ili } W_2 \in \mathcal{W}_2 \}$$

Primjetimo da proizvoljnu matricu $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ možemo napisati kao

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underbrace{\begin{pmatrix} a & -a \\ c & d \end{pmatrix}}_{\in \mathcal{W}_1} + \underbrace{\begin{pmatrix} 0 & b+a \\ 0 & 0 \end{pmatrix}}_{\in \mathcal{W}_2}$$

$\Rightarrow$ dimenzija od $\mathcal{W}_1 + \mathcal{W}_2$ je 4, pa za bazu možemo uzeti npr. $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

Ostalo je još da pronađemo bazu i dimenziju za $\mathcal{W}_1 \cap \mathcal{W}_2$.

Označimo sa $\mathcal{W}_1'$ i $\mathcal{W}_2'$ skup koordinata prostora $\mathcal{W}_1$ i $\mathcal{W}_2$ u odnosu na standardnu bazu. Tada

$$\mathcal{W}_1' = \left\{ \begin{pmatrix} x \\ -x \\ y \\ z \end{pmatrix} \middle| x,y,z \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \in \mathbb{R}^4 \middle| \alpha_1 + \alpha_2 = 0 \right\} =$$

$$= \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} \middle| \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} = \ker\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\mathcal{W}_2' = \left\{ \begin{pmatrix} a\\ b\\ -a\\ c\end{pmatrix} \,\middle|\, a,b,c\in\mathbb{R}\right\} = \left\{\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}\in\mathbb{R}^4 \,\middle|\, \alpha_1+\alpha_3=0\right\} =$$

$$= \left\{\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}\in\mathbb{R}^4 \,\middle|\, \begin{pmatrix}1&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right\} = \ker\left(\begin{bmatrix}1&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\right)$$

$$\mathcal{W}_1' \cap \mathcal{W}_2' = \left\{\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}\in\mathbb{R}^4 \,\middle|\, x_1+x_2=0 \;;\; x_1+x_3=0\right\} =$$

$$= \left\{\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}\in\mathbb{R}^4 \,\middle|\, \begin{pmatrix}1&1&0&0\\1&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right\} = \ker\Bigg(\underbrace{\begin{bmatrix}1&1&0&0\\1&0&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}}_{=A}\Bigg)$$

slobodni

Baza za $\ker A$ su vektori iz općeg rješenja sistema $Ax=0$.

rang $A = 2 =$ rang $\bar{A}$

$x_1+x_2=0 \Rightarrow x_2=-x_1 \qquad x_1=s$

$x_1+x_3=0 \Rightarrow x_3=-x_1 \qquad x_4=t$

$$\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix}s\\ -s\\ -s\\ t\end{pmatrix} = \begin{pmatrix}1\\ -1\\ -1\\ 0\end{pmatrix}s + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}t$$

Baza za $\mathcal{W}_1 \cap \mathcal{W}_2$ je $\left\{\begin{pmatrix}1&-1\\-1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$ a odatle slijedi da je $\dim(\mathcal{W}_1\cap\mathcal{W}_2)=2$.

Na kraju primijetimo da je zadovoljena formula

$$\dim(\mathcal{W}_1+\mathcal{W}_2) = \dim\mathcal{W}_1 + \dim\mathcal{W}_2 - \dim(\mathcal{W}_1\cap\mathcal{W}_2)$$

$$4 = 3 + 3 - 2$$

(#) Zadan je linearni operator $T: Mat_{2\times 2}(\mathbb{R}) \to Mat_{2\times 2}(\mathbb{R})$ sa

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-b & -a+b+2c \\ a-c-d & -a+2c+d \end{bmatrix}.$$

(a) Odredite po jednu bazu za $ker(T)$ i $im(T)$.

(b) Odredite matricu koordinata od T u odnosu na standardnu bazu prostora $Mat_{2\times 2}(\mathbb{R})$.

Rj.:

(a) Za jednostavniji pristup umjesto prostora $Mat_{2\times 2}(\mathbb{R})$ posmatrajmo prostor $\mathbb{R}^4$ i operator $T': \mathbb{R}^4 \to \mathbb{R}^4$ definisan sa

$$T'\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{pmatrix} a-b \\ -a+b+2c \\ a-c-d \\ -a+2c+d \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & 0 & -1 & -1 \\ -1 & 0 & 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = A \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$ker(T') = \{x \mid T'(x) = 0\} = \left\{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \,\middle|\, A\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \mathbf{0}\right\} = ker(A)$$

Znamo:

<u>Jezgro od A je opšte rješenje sistema $Ax = \mathbf{0}$.</u>

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ -1 & 0 & 2 & 1 & 0 \end{array}\right] \underset{\substack{III_v - I_v \\ IV_v + I_v}}{\overset{II_v + I_v}{\sim}} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & -1 & 2 & 1 & 0 \end{array}\right] \overset{II_v \leftrightarrow IV_v}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right] \overset{III_v + II_v}{\sim}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{array}\right] \overset{IV_v + III_v \cdot (-2)}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \ldots \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} a-d&=0 \\ b-d&=0 \\ c&=0 \end{aligned} \qquad \begin{pmatrix} x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix} s\\s\\0\\s\end{pmatrix} = \begin{pmatrix} 1\\1\\0\\1\end{pmatrix} s$$

$d \in \mathbb{R}$

Prema tome $\ker(T) = \text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ ← baza za $\ker(T)$

$$\text{im}(T) = \{ T(x) \mid x \in \mathbb{R}^4 \} = \left\{ A\begin{pmatrix} a\\b\\c\\d\end{pmatrix} \;\middle|\; \begin{pmatrix} a\\b\\c\\d\end{pmatrix} \in \mathbb{R}^4 \right\} = \text{im}(A)$$

Znamo:

<u>Generator skup za $\text{im}(A)$ su osnovne kolone u A.</u>

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & 0 & -1 & -1 \\ -1 & 0 & 2 & 1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \text{im}(T) = \text{span}\left\{ \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix} \right\}$$ ← baze za $\text{im}(T)$.

(b) Standardna baza za $\text{Mat}_{2\times 2}$ je $\left\{ \begin{bmatrix} 1&0\\0&0\end{bmatrix}, \begin{bmatrix} 0&1\\0&0\end{bmatrix}, \begin{bmatrix} 0&0\\1&0\end{bmatrix}, \begin{bmatrix} 0&0\\0&1\end{bmatrix} \right\}$

$$[T]_\varphi = \left(\left[T\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]_\varphi \;\; \left[T\begin{pmatrix}0&1\\0&0\end{pmatrix}\right]_\varphi \;\; \left[T\begin{pmatrix}0&0\\1&0\end{pmatrix}\right]_\varphi \;\; \left[T\begin{pmatrix}0&0\\0&1\end{pmatrix}\right]_\varphi \right)$$

$$T\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}1&-1\\1&-1\end{pmatrix}, \quad T\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}-1&1\\0&0\end{pmatrix}, \quad T\begin{pmatrix}0&0\\1&0\end{pmatrix} = \begin{pmatrix}0&2\\-1&2\end{pmatrix}, \quad T\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0\\-1&1\end{pmatrix}$$

$$[T]_\varphi = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & 0 & -1 & -1 \\ -1 & 0 & 2 & 1 \end{pmatrix}$$

Posmatrajmo realni unitarni prostor $\mathcal{P}_2$, gdje za polinome

$$p = p(x) = p_0 + p_1 x + p_2 x^2 \quad i \quad q = q(x) = q_0 + q_1 x + q_2 x^2$$

je definisan unutrašnji proizvod na sljedeći način

$$\langle p, q \rangle = p_0 q_0 + p_1 q_1 + p_2 q_2 .$$

Provjeriti da li su polinomi

$$u_1 = 3 + 4x + 5x^2, \quad u_2 = 9 + 12x + 5x^2, \quad u_3 = 1 - 7x + 25x^2$$

linearno nezavisni u $\mathcal{P}_2$, pa pomoću njih formirati ortonormiranu bazu za $\mathcal{P}_2$.

Rj.

$$\alpha u_1 + \beta u_2 + \gamma u_3 = 0$$

$$\begin{aligned} 3\alpha + 9\beta + \gamma &= 0 \\ 4\alpha + 12\beta - 7\gamma &= 0 \\ 5\alpha + 5\beta + 25\gamma &= 0 \end{aligned} \quad \Rightarrow \quad \begin{vmatrix} 3 & 9 & 1 \\ 4 & 12 & -7 \\ 5 & 5 & 25 \end{vmatrix} = -250 \neq 0$$

Vektori (polinomi) su linearno nezavisni.

Primjenimo Gram-Schmidt-ovu proceduru na ovu bazu. Prisjetimo se (jednog od načina)

$$\underline{v_1 = u_1}$$

$$\underline{v_2 = u_2 - \frac{\langle v_1, u_2 \rangle}{\|v_1\|^2} v_1}$$

$$\underline{v_3 = u_3 - \frac{\langle v_1, u_3 \rangle}{\|v_1\|^2} v_1 - \frac{\langle v_2, u_3 \rangle}{\|v_2\|^2} v_2}$$

i time ćemo dobiti ortogonalne vektore $\{v_1, v_2, v_3\}$

Pa krenimo redom

$$v_1 = u_1 = 3 + 4x + 5x^2$$

$$V_2 = U_2 - \frac{\langle V_1, U_2 \rangle}{\|V_1\|^2} = (9+12x+5x^2) - \frac{\langle 3+4x+5x^2,\ 9+12x+5x^2 \rangle}{\|3+4x+5x^2\|^2}(3+4x+5x^2)$$

$$= (9+12x+5x^2) - \frac{100}{50}(3+4x+5x^2) = (9+12x+5x^2) + (-6-8x-10x^2) =$$

$$= 3+4x-5x^2$$

$$V_3 = U_3 - \frac{\langle V_1, U_3 \rangle}{\|V_1\|^2} V_1 - \frac{\langle V_2, U_3 \rangle}{\|V_2\|^2} V_2 =$$

$$= (1-7x+25x^2) - \frac{\langle 3+4x+5x^2,\ 1-7x+25x^2 \rangle}{\|3+4x+5x^2\|^2}(3+4x+5x^2)$$

$$- \frac{\langle 3+4x-5x^2,\ 1-7x+25x^2 \rangle}{\|3+4x-5x^2\|^2}(3+4x-5x^2) =$$

$$= (1-7x+25x^2) - \frac{100}{50}(30+4x+5x^2) + \frac{150}{50}(3+4x-5x^2)$$

$$= (1-7x+25x^2) + (-6-8x-10x^2) + (9+12x-15x^2) = 4-3x+0x^2$$

Sad nije teško provjeriti da su tri dobijena polinoma

$$V_1 = 3+4x+5x^2, \quad V_2 = 3+4x-5x^2, \quad V_3 = 4-3x+0x^2$$

parno ortogonalni, tako da je $\{V_1, V_2, V_3\}$ ortogonalna baza za $\mathcal{P}_2$. Normalizirajući svaku od ova tri polinoma, dobićemo odgovarajuću ortonormiranu bazu

$$\left\{ \frac{3}{\sqrt{50}} + \frac{4}{\sqrt{50}}x + \frac{5}{\sqrt{50}}x^2,\ \frac{3}{\sqrt{50}} + \frac{4}{\sqrt{50}}x - \frac{5}{\sqrt{50}}x^2,\ \frac{4}{5} - \frac{3}{5}x + 0x^2 \right\}.$$

(#) Neka je $\mathcal{M}$ podprostor unitarnog prostora $Mat_{2\times2}(\mathbb{R})$ generisan matricama $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ i $\begin{bmatrix}2&1\\0&0\end{bmatrix}$. Odredite jednu bazu za ortogonalni kompliment od $\mathcal{M}$, te prikažite matricu $X=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ u obliku $X=Y_1+Y_2$, gdje je $Y_1\in\mathcal{M}$, a $Y_2\in\mathcal{M}^\perp$. (Standardni skalarni proizvod u $Mat_{2\times2}(\mathbb{R})$ je $\langle A,B\rangle=trag(AB^T)$).

Rj:

Prisjetimo se definicije ortogonalnog komplementa od $\mathcal{M}$:

$$\underline{\mathcal{M}^\perp=\{x\in V \mid \langle m,x\rangle=0 \text{ za } \forall m\in\mathcal{M}\}}$$

U našem slučaju

$$\mathcal{M}=span\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}2&1\\0&0\end{bmatrix}\right\}.$$ Neka je $M=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ i $N=\begin{bmatrix}2&1\\0&0\end{bmatrix}$.

Odredimo matricu $C=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ t.d. $\langle M,C\rangle=0$ i
$\langle N,C\rangle=0$

$$\left.\begin{aligned}\langle M,C\rangle&=trag\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}a&c\\b&d\end{bmatrix}\right)=trag\left(\begin{bmatrix}a&c\\b&d\end{bmatrix}\right)=a+d \text{ i}\\ \langle N,C\rangle&=trag\left(\begin{bmatrix}2&1\\0&0\end{bmatrix}\begin{bmatrix}a&c\\b&d\end{bmatrix}\right)=trag\left(\begin{bmatrix}2a+b&2c+d\\0&0\end{bmatrix}\right)=2a+b\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow \begin{aligned}a+d&=0\\2a+b&=0\end{aligned} \Rightarrow \begin{array}{c}\begin{matrix}a&b&c&d&\end{matrix}\\ \left[\begin{array}{cccc|c}1&0&0&1&0\\2&1&0&0&0\end{array}\right]\end{array} \overset{II_v-I_v\cdot 2}{\sim} \left[\begin{array}{cccc|c}1&0&0&1&0\\0&1&0&-2&0\end{array}\right] \Rightarrow \begin{aligned}a&=-d\\b&=2d\end{aligned}$$

$rang A=rang\bar{A}=2<4 \Rightarrow$ 2 promjenjive uzimamo proizvoljno $\begin{matrix}d=t\\c=s\end{matrix}$

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}-t&2t\\s&t\end{bmatrix}=\begin{bmatrix}0&0\\1&0\end{bmatrix}s+\begin{bmatrix}-1&2\\0&1\end{bmatrix}t$$

Prema tome $\mathcal{M}^{\perp} = \text{span}\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \right\}$.

Baza za ortogonalni komplement je $\left\{ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \right\}$

Ostalo je još da odredimo α, β, γ i δ t.d.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \beta \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \delta \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \alpha + 2\beta \quad - \delta &= 1 \\ \beta \quad + 2\delta &= 0 \\ \gamma &= 0 \\ \alpha \quad + \delta &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right] \sim \dots \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1/6 \\ 0 & 1 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/6 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{5}{6} & \frac{1}{3} \\ 0 & \frac{1}{6} \end{bmatrix}}_{\in \mathcal{M}} + \underbrace{\begin{bmatrix} \frac{1}{6} & -\frac{1}{3} \\ 0 & -\frac{1}{6} \end{bmatrix}}_{\in \mathcal{M}^{\perp}}$$

$\frac{1}{6} + \frac{2 \cdot 2}{3 \cdot 2} = \frac{5}{6}$